

Describing the set of words generated by interval exchange transformation

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1 Introduction

Methods of symbolic dynamics are rather useful in the study of combinatorial properties of words, investigation of problems of number theory and theory of dynamical systems. Let M be a compact metric space, $U \subset M$ be its open subspace, $f : M \rightarrow M$ be a homeomorphism of the compact into itself, and $x \in M$ be an initial point. With the sequence of iterations, one can associate an infinite binary word

$$w_n = \begin{cases} a, & f^{(n)}(x_0) \in U \\ b, & f^{(n)}(x_0) \notin U \end{cases}$$

which is called the *evolution* of point x_0 . Symbolic dynamics investigates the interrelation between the properties of the dynamical system (M, f) and the combinatorial properties of the word W_n . For words over alphabets which comprise more symbols, several characteristic sets should be considered: U_1, \dots, U_n .

By the *direct problem* of symbolic dynamics we mean the study of combinatorial properties of the words generated by a given dynamical system; the *inverse problem* of symbolic dynamics refers to the investigation of the properties of the dynamical system, i.e., the properties of the compact set M and transform f , by the combinatorial properties of the word W .

Inverse problems of symbolic dynamics related to the unipotent transformation of a torus were studied in paper [2].

Problems (both direct and inverse) related to the rotation of a circle bring about a class of words which are called *Sturmian words*. Sturmian words are infinite words over a binary alphabet which contain exactly $n + 1$ different subwords (factors) of length n for any $n \geq 1$. The following classical result is widely known.

Theorem 1.1 (Equivalence theorem ([21],[20]).) *Let W be an infinite recurrent word over the binary alphabet $A = \{a, b\}$. The following conditions are equivalent:*

1. *The word W is a Sturmian word, i.e., for any $n \geq 1$, the number of different subwords of length n that occur in W is equal to $T_n(W) = n + 1$.*
2. *The word is not periodic and is balanced, i.e., any two subwords $u, v \subset W$ of the same length satisfy the inequality $||v|_a - |u|_a| \leq 1$, where $|w|_a$ denotes the number of occurrences of symbol a in the word w .*
3. *The word $W = (w_n)$ is a mechanical word with irrational α , which means that there exist an irrational α , $x_0 \in [0, 1]$, and interval $U \subset \mathbb{S}^1$, $|U| = \alpha$, such that the following condition holds:*

$$w_n = \begin{cases} a, & T_\alpha^n(x_0) \in U \\ b, & T_\alpha^n(x_0) \notin U \end{cases}$$

There are several different ways of generalizing Sturmian words.

First, one can consider *balanced* words over an arbitrary alphabet. Balanced nonperiodic words over an n -letter alphabet were studied in paper [17] and later in [18]. In papers [3] and [5], a dynamical system that generates an arbitrary nonperiodic balanced word was constructed.

Second, generalization may be formulated in terms of the *complexity function*. Complexity function $T_W(n)$ presents the number of different subwords of length n in the word W . Sturmian words satisfy the relation $T_W(n+1) - T_W(n) = 1$ for any $n \geq 1$. Natural generalizations of Sturmian words are words with minimal growth, i.e., words over a finite alphabet that satisfy the relation $T_W(n+1) - T_W(n) = 1$ for any $n \geq k$, where k is a positive integer. Such words were described in terms of rotation of a circle in paper [6]. Note also that words whose growth function satisfies the relation $\lim_{n \rightarrow \infty} T(n)/n = 1$ were studied in paper [7].

Words with complexity function $T_W(n) = 2n + 1$ were studied by P. Arnoux and G. Rauzy ([8, 22, 23]), words with growth function $T_W(n) = 2n + 1$ were investigated by G. Rote [24]. Consideration of the general case of words with linear complexity function involves the study of words generated by interval exchange transformations. The problem of description of such words was posed by Rauzy [23]. Words with linear growth of the number of subwords were also studied by the school of V. Berthé, S. Ferenczi, and Luca Q. Zamboni ([14], [9]). They also investigate combinatorial sequences related with interval exchange transformations. Paper [16] contains description of words generated by three-interval exchange transformations; paper [14] contains description of words generated by symmetric interval exchange transformations (such transformations are closely related to multi-dimensional continued fractions, and this relation looks extremely interesting). More precisely, they describe a combinatorial algorithm for generating the symbolic sequences which code the orbits of points under an interval exchange transformation on k intervals, using the symmetric permutation $i \rightarrow k - i + 1$ ([14]).

More general result was obtained in the work [15]. In this paper give a complete characterization of those sequences of subword complexity $(k-1)n + 1$ which are natural codings of orbits of k -interval exchange transformations, (or, equivalently, interval exchange transformation, satisfying i.d.o.c. condition).¹

Interval exchange transformations T satisfies the *infinite distinct orbit condition* (or *i.d.o.c.* for short) if the $k-1$ negative trajectories $\{T^{-n}(x_i)\}_{n \geq 0}, (1 \leq i \leq k)$, of the discontinuities of T are infinite disjoint sets. The main result of paper [15] (which is independent of our result) is following:

Theorem 1.2 *A minimal sequence W is the natural coding of a k -interval exchange transformation, defined by permutations (π_0, π_1) such that $\pi_0^{-1}(\{1, \dots, j\}) \neq \pi_1^{-1}(\{1, \dots, j\})$ for every $1 \leq j \leq k-1$, and satisfying the i.d.o.c. condition, if and only if the words of length one occurring in W are $F_1 = \{1, \dots, k\}$ and it satisfies the following conditions:*

1. *If w is any word occurring in W , $A(w)$, (resp. $D(w)$), the set of all letters x such that xw , (resp. wx), occurs in W , is an interval for the order of π_1 , resp. π_0 ,*
2. *If $x \in A(w)$, $y \in A(w)$, $x \leq y$ for the order of π_1 , $z \in D(xw)$, $t \in D(yw)$, then $z \leq t$ for the order of π_0 ,*
3. *If $x \in A(w)$ and $y \in A(w)$ are consecutive in the order of π_1 , $D(xw) \cap D(yw)$ is a singleton.*

In this paper we study words generated by general piecewise-continuous transformation of the interval. Further we prove equivalence set words generated by piecewise-continuous

¹This result which is close to ours, was noticed to us by L.Zamboni, in order to mention that and make some other corrections we replaced our paper to new version

transformation and words generated by interval exchange transformation. This method get capability of descriptions of the words generated by arbitrary interval exchange transformation.

This work is targeted to the following

Inverse problem: *Which conditions should be imposed on a uniformly recurrent word W in order that it be generated by a dynamical system of the form (I, T, U_1, \dots, U_k) , where I is the unit interval and T is the interval exchange transformation?*

The answer to this question is given in terms of the evolution of the *labeled Rauzy graphs* of the word W . The *Rauzy graph* of order k (the k -graph) of the word W is the directed graph whose vertices biuniquely correspond to the factors of length k of the word W and there exists an arc from vertex A to vertex B if and only if W has a factor of length $k + 1$ such that its first k letters make the subword that corresponds to A and the last k symbols make the subword that corresponds to B . By the *follower* of the directed k -graph G we call the directed graph $\text{Fol}(G)$ constructed as follows: the vertices of graph $\text{Fol}(G)$ are in one-to-one correspondence with the arcs of graph G and there exists an arc from vertex A to vertex B if and only if the head of the arc A in the graph G is at the notch end of B . The $(k + 1)$ -graph is a subgraph of the follower of the k -graph; it results from the latter by removing some arcs. Vertices which are tails of (or heads of) at least two arcs correspond to *special factors* (see Section 2); vertices which are heads and tails of more than one arc correspond to *bispecial factors*. The sequence of the Rauzy k -graphs constitutes the *evolution* of the Rauzy graphs of the word W . The Rauzy graph is said to be *labeled* if its arcs are assigned letters l and r and some of its vertices (perhaps, none of them) are assigned symbol “-”.

The *follower* of the labeled Rauzy graph is the directed graph which is the follower of the latter (considered a Rauzy graph with the labeling neglected) and whose arcs are labeled according to the following rule:

1. Arcs that enter a branching vertex should be labeled by the same symbols as the arcs that enter any left successor of this vertex;
2. Arcs that go out of a branching vertex should be labeled by the same symbols as the arcs that go out of any right successor of this vertex;
3. If a vertex is labeled by symbol “-”, then all its right successors should also be labeled by symbol “-”.

In terms of Rauzy labeled graphs we define the *asymptotically correct* evolution of Rauzy graphs, i.e., we introduce rules of passing from k -graphs to $(k + 1)$ -graphs. Namely, the evolution is said to be *correct* if, for all $k \geq 1$, the following conditions hold when passing from the k -graph G_k to the $(k + 1)$ -graph G_{k+1} :

1. The degree of any vertex is at most 2, i.e., it is incident to at most two incoming and outgoing arcs;
2. If the graph contains no vertices corresponding to bispecial factors, then G_{n+1} coincides with the follower $D(G_n)$;
3. If the vertex that corresponds to a bispecial factor is not labeled by symbol “-”, then the arcs that correspond to forbidden words are chosen among the pairs lr and rl ;

4. If the vertex is labeled by symbol “–”, then the arcs to be deleted should be chosen among the pairs ll or rr .

The evolution is said to be *asymptotically correct* if this condition is valid for all k beginning with a certain $k = K$. The *oriented* evolution of the graphs means that there are no vertices labeled by symbol “–”. The main result of this work consists in the description of infinite words generated by interval exchange transformations (and answers a Rauzy question [23]):

Main theorem. *A uniformly recurrent word W*

1. *is generated by an interval exchange transformation if and only if the word is provided with the asymptotically correct evolution of the labeled Rauzy graphs;*
2. *is generated by an orientation-preserving interval exchange transformation if and only if the word is provided with the asymptotically correct oriented evolution of the labeled Rauzy graphs.*

We have no restriction on the endpoint orbit. In special case of asymptotical subword growth of $T_W(n) = n + \text{const}$ for all $n > n_0$ we get an generalization of theorem 1.1, i.e. description of all u.r. words with such growth property. The description of all (not nesessary u.r.) superwords such that $T_W(n) = n + \text{const}$ for all $n > n_0$ see in [6]. Note also that in all previous studies which is known for us interval exchange transformations are defined to be orientation preserving.

The paper is organized as follows: in Section 2 we formulate the main definitions and facts about uniformly recurrent words, Rauzy graphs, and words generated by dynamical systems. In Section 3 we prove a theorem about necessary conditions for a word to be generated by interval exchange transformation. The next two sections are devoted to the proof of the sufficiency of these conditions. In Section 4 we prove that these conditions are sufficient for the word to be generated by a piecewise-continuous interval transformation. Finally, in Section 5 we prove that the sets of uniformly recurrent words generated by piecewise-continuous interval transformations and by the interval exchange transformation are equivalent.

2 Main constructions and definitions

2.1 Complexity function, special factors, and uniformly recurrent words

In this section we define the basic notions of combinatorics of words. By L we denote a finite alphabet, i.e., a nonempty set of elements (symbols). We use the notation A^+ for the set of all finite sequences of symbols or *words*.

A finite word can always be uniquely represented in the form $w = w_1 \cdots w_n$, where $w_i \in A$, $1 \leq i \leq n$. The number n is called the *length* of word w ; it is denoted by $|w|$.

The set A^+ of all finite words over A is a simple semigroup with concatenation as semigroup operation.

If element Λ (the empty word) is included in the set of words, then this is actually the free monoid A^* over A . By definition the length of the empty word is $|\Lambda| = 0$.

A word u is a *subword* (or *factor*) of a word w if there exist words $p, q \in A^+$ such that $w = puq$.

Denote the set of all factors (both finite and infinite) of a word W by $F(W)$. Two infinite words W and V over alphabet A are said to be *equivalent* if $F(W) = F(V)$.

We say that symbol $a \in A$ is a *left* (accordingly, *right*) *extension* of factor v if av (accordingly, va) belongs to $F(W)$. A subword v is called a *left* (accordingly, *right*) *special factor* if it possesses at least two left (right) extensions. A subword v is said to be *bispecial* if it is both a left and right special factor at the same time. The number of different left (right) extensions of a subword is called the *left* (*right*) *valence* of this subword.

A word W is said to be *recurrent* if each its factor occurs in it infinitely many times (in the case of a doubly-infinite word, each factor occurs infinitely many times in both directions). A word W is said to be *uniformly recurrent* or (*u.r. word*) if it is recurrent and, for each its factor v , there exists a positive integer $N(v)$ such that, for any subword u of length at least $N(v)$ of the word W , factor v occurs in u as a subword.

Below we formulate several theorems about u.r. words, which will be needed later. The proof of these theorems can be found in monograph [1].

Theorem 2.1 *The following two properties of an infinite word W are equivalent:*

- a) *For any k there exists $N(k)$ such that any segment of length k of the word W occurs in any segment of length $N(k)$ of the word W ;*
- b) *If all finite factors of a word V are at the same time finite factors of a word W , then all finite factors of the word W are also finite factors of the word V .*

Theorem 2.2 *Let W be an infinite word. Then there exists a uniformly recurrent word \widehat{W} all of whose factors are factors of W .* □

One can consider the action of the shift operator τ on the set of infinite words. The *Hamming distance* between words W_1 and W_2 is the quantity $d(W_1, W_2) = \sum_{n \in \mathbb{Z}} \lambda_n 2^{-|n|}$, where $\lambda_n = 0$ if symbols at the n -th positions of the words are the same and $\lambda_n = 1$, otherwise.

An *invariant subset* is a subset of the set of all infinite words which is invariant under the action of τ . A *minimal closed invariant set*, or briefly, *m.c.i.s.*, is a closed (with respect to the Hamming metric introduced above) invariant subset which is nonempty and contains no closed invariant subsets except for itself and the empty subset.

Theorem 2.3 (Properties of closed invariant sets) *The following properties of a word W are equivalent:*

- 1. *W is a uniformly recurrent word;*
- 2. *The closed orbit of W is minimal and is a m.c.i.s.*

Theorem 2.4 *Let W be a uniformly recurrent nonperiodic infinite word. Then*

- 1. *All the words that are equivalent to W are u.r. words; the set of such words is uncountable;*
- 2. *There exist distinct u.r. words $W_1 \neq W_2$ which are equivalent to the given word and can be written as $W_1 = UV_1$, $W_2 = UV_2$, where U is a left-infinite word and $V_1 \neq V_2$ are right-infinite words.*

2.2 Rauzy graphs

It is convenient to describe a word W using the *subword graphs* or (Rauzy graphs). They were introduced by Rauzy [22] in the following way: the k -graph of the word W is the directed graph whose vertices biuniquely correspond to subwords of length k of the word W and there is an arc from vertex A to vertex B if W contains a subword of length $k + 1$ such that the first k symbols of it make a subword that corresponds to A and the last k symbols make a subword that corresponds to B . Thus, the arcs of the k -graph are in one-to-one correspondence with the $(k + 1)$ -factors of the word W .

It is clear that, in the k -graph G of the word W , vertices which are tails (accordingly, heads) of more than one arc correspond to right special words. Such vertices will be called *crotches*. Graph G is said to be *strongly connected* if it contains a directed path from any vertex to any other one.

The *follower* of the directed graph G is the directed graph $\text{Fol}(G)$ constructed in the following way: the vertices of the graph G biuniquely correspond to the arcs of the G and there is an arc from vertex A to vertex B if the head of the arc A in the graph G is at the notch end of B .

The connectivity of the Rauzy graphs is naturally related to the recurrence of the corresponding word. Namely, the following assertion is valid.

Proposition 2.5 *Let W be a (semi)infinite word. The following conditions are equivalent:*

1. *The word W is recurrent;*
2. *For any k the corresponding k -graph of the word W is strongly connected;*
3. *Any factor of W occurs at least twice;*
4. *Any factor can be extended to the left.*

Let us introduce the notion of the *labeled Rauzy graph*. A Rauzy graph is said to be *labeled* if

1. the arcs of any crotch are assigned symbols l (“left”) and r (“right”);
2. some vertices are assigned symbol “—”.

The *follower* of the labeled Rauzy graph is the directed graph which is the follower of the latter (considered a Rauzy graph with the labeling neglected) and whose arcs are labeled according the following rule:

1. Arcs that enter a crotch should be labeled by the same symbols as the arcs that enter any left successor of this vertex;
2. Arcs that go out of a crotch should be labeled by the same symbols as the arcs that go out of any right successor of this vertex;
3. If a vertex is labeled by symbol “—”, then all its right successors should also be labeled by symbol “—”.

2.3 Words generated by dynamical systems

Let M be a compact metric space, $U \subset M$ be its open subset, $f : M \rightarrow M$ be a homeomorphism of the compact space into itself, and $x \in M$ be an initial point.

With the sequence of iterations, one can associate an infinite binary word

$$w_n = \begin{cases} a, & f^{(n)}(x_0) \in U \\ b, & f^{(n)}(x_0) \notin U \end{cases}$$

which is called the *evolution* of point x_0 . Symbolic dynamics investigates the interrelation between the properties of the dynamical system (M, f) and the combinatorial properties of the word W_n .

For words over alphabets which comprise more symbols, several characteristic sets should be considered: U_1, \dots, U_n .

Note that the evolution of the point is correctly defined only when the trajectory of the point does not pass through the boundary of the characteristic sets $\partial U_1, \partial U_2, \dots$

In order to consider the trajectory of an arbitrary point, let us introduce the notion of *essential evolution*.

Definition 2.6 *A finite word v^f is called essential finite evolution of point x^* if any neighborhood of point x^* contains an open set V such that any point $x \in V$ possesses the evolution v^f . An infinite word W is called essential evolution of point x^* if any its initial subword is an essential finite evolution of point x^* .*

When there is no risk of ambiguity, we say *evolution of the point* meaning the essential evolution. Note that a point can have several essential evolutions.

Proposition 2.7 ([2]) *Let V be a finite word. Then the set of points with the finite essential evolution V is closed. A similar assertion is true for an infinite word W .*

2.4 Correspondence between words and partitions of a set

Now let us consider the correspondence between words and subsets of M . It follows from the construction that, if the initial point belongs to the set U_i , then its evolution begins with symbol a_i . Consider the images of the sets U_i under the mappings $f^{(-1)}, f^{(-2)}, \dots$, $n \in \mathbb{N}$.

It is clear that, if the point belongs to the set

$$f^{(-n)}(U_{i_n}) \cap f^{-(n-1)}(U_{i_{n-1}}) \cap \dots \cap f^{(-1)}(U_{i_1}) \cap U_{i_0},$$

then the evolution begins with the word $a_{i_0}a_{i_1} \dots a_{i_n}$.

Accordingly, the number of different essential evolutions of length $n+1$ is equal to the number of partitions of the set M into nonempty subsets by the boundaries of the sets ∂U_i and their images under the mappings $f^{(-1)}, f^{(-2)}, \dots, f^{(-n)}$.

Remark. The number of finite essential evolutions is directly related to the topological dimension of the set M . For instance, it is clear that, if M is homeomorphic to a segment or a circle, then one point can belong to the boundary of at most two open subsets of M and, accordingly, can have only two essential evolutions. If M is homeomorphic to a part of the plane \mathbb{R}^2 then there can be arbitrarily many essential evolutions.

2.5 Interval exchange transformations

Interval exchange transformation is a natural generalization of the shift of a circle: in the case of the partition of a circle into arcs of length α and $1 - \alpha$ and a shift of α , this transformation coincides with the two-interval exchange transformation.

In addition, interval exchange transformation is rather important in ergodic theory, theory of dynamical systems, and number theory.

Let us consider the general case:

Suppose that the closed interval $[0, 1]$ is partitioned into half-open intervals of lengths $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\sigma \in S_n$ is a permutation of the set $\{1, 2, \dots, n\}$.

The intervals of the partition can be represented in terms of the lengths given above:

$$X_i = \left[\sum_{j < i} \lambda_j, \sum_{j \leq i} \lambda_j \right).$$

The interval exchange transformation rearranges the intervals (X_1, X_2, \dots, X_n) of the partition; as a result, we obtain a new partition

$$(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(n)}).$$

In the orientation-preserving case, transformation T associates each point $x \in X_i$:

$$T(x) = x + a_i,$$

where

$$a_i = \sum_{k < \sigma^{-1}(i)} \lambda_{\sigma(k)} - \sum_{k < i} \lambda_k.$$

If the transformation inverts an interval, then, in addition, all points are symmetrically reflected with respect to the midpoint of this segment.

Definition 2.8 *Interval exchange transformation T is said to be regular if, for any point a_i , where $X_i = [a_i, a_{i+1})$, we have $T^n(a_i) \neq a_j$.*

The result formulated below is rather important (see [16]):

Theorem 2.9 *An interval exchange transformation is regular if and only if the trajectory of any point is dense everywhere in $[0, 1]$.*

Properties of words generated by interval exchange transformations are investigated using the same methods as in the case of the shift of the unit circle. Here, the main approach consists in considering the negative orbits of the ends of the exchanged intervals

$$0 = a_1 < a_2 < \dots < a_{k+1} = 1,$$

where

$$X_i = [a_i, a_{i+1}), (i \in \{1, \dots, k\}).$$

Denote the set of ends of the exchanged intervals $\{a_i | 1 \leq i \leq k+1\}$ by X^1 . A word w of length n is a subword of the evolution of point x , i.e., the infinite word $U(x)$, if and only if there exists an interval $I_w \subset [0, 1]$ and point $y \in I_w$ such that the word w is the concatenation of the symbols:

$$\mathcal{I}(x)\mathcal{I}(T(x))\cdots\mathcal{I}(T^{n-1}(x)) = w,$$

where $\mathcal{I}(x) = a_i \in A$ if and only if $x \in X_i$.

Proposition 2.10 *Let T be a regular k -interval exchange transformation. Then the evolution $U(x)$ of any point x has the complexity function $T_{U(x)}(n) = n(k-1) + 1$ for any $n \in \mathbb{N}$.*

3 Necessary conditions for a word to be generated by an interval exchange transformation

At the first stage we formulate necessary conditions for a word to be generated by an interval exchange transformation. Let the word W be the evolution of point $x \in [0, 1]$ for a k -interval exchange transformation and characteristic sets U_1, U_2, \dots, U_n . Each characteristic set U_i is a union of several disjoint open or half-open intervals.

As was already shown in Section 2.4, subwords of length k are in one-to-one correspondence with the k -partitions of the characteristic sets. Since a boundary point of a one-dimensional set can belong to the boundary of only two sets, a k -subword of the word W can have at most two extensions. We obtain the first necessary condition for a word to be generated by interval exchange transformation:

Proposition 3.1 *Let the word W be generated by interval exchange transformation. Then, for a certain N , all special subwords of length at least N should have valence 2.*

This condition is similar to the condition that, starting from a certain N , all k -graphs of the word W ($k \geq N$) should have all incoming and outgoing crotches of degree 2.

Now let us derive conditions in terms of Rauzy graphs. Suppose that the recurrent word W has the growth function $F_W(n) = Kn + L$ for $n > N$ and is generated by interval exchange transformation. Consider the evolution of Rauzy k -graphs of the word W starting with $k \geq N+1$. As has already been demonstrated, all incoming and outgoing crotches have degree 2; therefore, all k -graphs have exactly K incoming and K outgoing crotches.

In the minimal case described in the previous section, where $F_W(n) = n + L$, Rauzy graphs have exactly one incoming and one outgoing crotch. If the incoming crotch coincides with the outgoing one, then the choice of the arc to be deleted from the follower $D(G)$ is uniquely determined by the condition of strong connectivity of the graph.

Let us thoroughly investigate a more interesting case, where graphs of words contain more than one crotch. There are several possible situations that should be considered when passing from graph G_n to G_{n+1} :

1. Graph G_n contains no linked cycles (i.e., there are no incoming crotches that are at the same time outgoing crotches). In this case, graph G_{n+1} coincides with the follower $D(G_n)$.
2. Graph G_n contains one crotch which is at the same time an incoming and outgoing crotch. In this case, the follower graph $D(G_n)$ has three crotches, because one crotch has been cloned. Therefore, the graph G_{n+1} is obtained from the follower $D(G_n)$ by means of deleting one arc which corresponds to the minimal non-occurring word.

3. The graph contains two or more crotches which are at the same time incoming and outgoing crotches. Then the graph G_{n+1} is obtained from $D(G_n)$ by means of deleting two or more arcs which correspond to the minimal non-occurring words.

Since the word W is recurrent, it follows from Proposition 2.5 that, as the arcs are deleted, the graph should remain strongly connected, i.e., it should contain a directed path from any vertex to any other one.

Consider the second case in more detail. Suppose that G_k contains one double crotch. This means that W contains exactly one bispecial subword w of length k . Hence, there exist $a_i, a_j, a_k, a_l \in A$ such that a_iw, a_jw, wa_k , and wa_l are factors of W . Then the $(k+1)$ -graph G_{k+1} is obtained from the follower by means of deleting an arc that corresponds to one of the four words: a_iwa_k , a_iwa_l , a_jwa_k , or a_jwa_l . Consider the interval which is the characteristic set for the word, $I_w = [x_w, y_w]$.

Since w is a right special word, we have $I_w \subset T^{-1}(I_{a_k} \cup I_{a_l})$; since w is a left special word, we have $I_w \subset T(I_{a_i} \cup I_{a_j})$.

Suppose that point $A \in [0, 1]$ partitions I_w into two intervals whose images lie in I_{a_k} and I_{a_l} , respectively, and point $B \in [0, 1]$ partitions it into intervals whose preimages lie in I_{a_i} and I_{a_j} , respectively.

The choice of the minimal non-occurring word (and, hence, the arc to be deleted) is determined by the mutual location of points A and B and by whether the mapping preserves orientation on these sets or they are reversed. There are 8 cases, which are divided into four pairs which correspond to similar sets of words:

1. $B < A$, $T^{-1}([x_w, B]) \subset I_{a_i}$, $T([x_w, A] \subset I_{a_k})$
2. $B < A$, $T^{-1}([x_w, B]) \subset I_{a_j}$, $T([x_w, A] \subset I_{a_k})$
3. $B < A$, $T^{-1}([x_w, B]) \subset I_{a_i}$, $T([x_w, A] \subset I_{a_l})$
4. $B < A$, $T^{-1}([x_w, B]) \subset I_{a_j}$, $T([x_w, A] \subset I_{a_l})$
5. $B > A$, $T^{-1}([x_w, B]) \subset I_{a_i}$, $T([x_w, A] \subset I_{a_k})$
6. $B > A$, $T^{-1}([x_w, B]) \subset I_{a_j}$, $T([x_w, A] \subset I_{a_k})$
7. $B > A$, $T^{-1}([x_w, B]) \subset I_{a_i}$, $T([x_w, A] \subset I_{a_l})$
8. $B > A$, $T^{-1}([x_w, B]) \subset I_{a_j}$, $T([x_w, A] \subset I_{a_l})$

Cases 1 and 5 correspond to the forbidden word a_iwa_k .

Cases 2 and 6 correspond to the forbidden word a_iwa_l .

Cases 3 and 7 correspond to the forbidden word a_jwa_k .

Cases 4 and 8 correspond to the forbidden word a_jwa_l .

Two pairs of cases correspond to simultaneous reversal or preservation of orientation of the mapping on characteristic intervals; two other pairs correspond to opposite orientations.

In the case where the interval exchange transformation preserves orientation, we have only two possibilities for the arc to be deleted.

If the mapping is allowed to reverse the intervals in the process of the interval exchange transformation, all four cases are possible.

Let us introduce the notion of the *labeled Rauzy graph*. A Rauzy graph is said to be *labeled* if

1. The arcs of any crotch are assigned symbols l (“left”) and r (“right”);
2. Some vertices are assigned symbol “–”.

The *follower* of the labeled Rauzy graph is the directed graph which is the follower of the latter (considered a Rauzy graph with the labeling neglected) and whose arcs are labeled according the following rule:

1. Arcs that enter a crotch should be labeled by the same symbols as the arcs that enter any left successor of this vertex;
2. Arcs that go out of a crotch should be labeled by the same symbols as the arcs that go out of any right successor of this vertex;
3. If a vertex is labeled by symbol “–”, then all its right successors should also be labeled by symbol “–”.

Remark. Now let us explain the meaning of the labeled Rauzy graph. Let the arcs of the incoming crotch correspond to a_i and a_j and symbols l and r correspond to the left and right set in the pair $(T(I_{a_i}), T(I_{a_j}))$. If symbols a_k and a_l correspond to the arcs of the outgoing crotch, then symbols l and r appear in accordance with the “left–right” order in the pair (I_{a_k}, I_{a_l}) . A vertex is assigned symbol “–” if the characteristic set that corresponds to it belongs to an interval which is reversed in the process of interval exchange transformation.

Below, we give a condition for passing from graph G_n to G_{n+1} .

Proposition 3.2 *1. If the graph contains no double crotches that correspond to bispecial factors, then, when passing from G_n to G_{n+1} , we have $G_{n+1} = D(G_n)$.*

- 2. If the vertex that corresponds to a bispecial word is not labeled by symbol “–”, then the arcs that correspond to the forbidden words are chosen among the pairs lr and rl .*
- 3. If a vertex is labeled by symbol “–”, then the arcs to be deleted should be chosen among the pair ll or rr .*

The evolution of labeled Rauzy graphs is said to be *correct* if **Rules 1** and **2** are complied with by all graphs in the evolution starting from G_1 ; the evolution is said to be *asymptotically correct* if **Rules 1** and **2** are complied with starting from a certain G_n . We say that the evolution of labeled Rauzy graphs is *oriented* if the k -graphs contain no vertices labeled by symbol “–”.

The definition of the asymptotically correct evolution of Rauzy graphs allows us to formulate the conditions for a word to be generated by interval exchange transformation.

Proposition 3.3 *A uniformly recurrent word W*

- 1. is generated by interval exchange transformation if the word is provided with the asymptotically correct evolution of labeled Rauzy graphs;*
- 2. is generated by orientation-preserving interval exchange transformation if the word is provided with the asymptotically correct oriented evolution of labeled Rauzy graphs.*

Our main result consists in replacing “if” with “if and only if” in the proposition formulated above.

3.1 Construction of the dynamical system

Let us demonstrate that conditions of Theorem 3.3 are sufficient for the word to be generated by interval exchange transformation. First, we show that the word W that satisfies the hypotheses of Theorem 3.3 can be the evolution of a certain point under the following piecewise-continuous transformation of the segment $T : I \rightarrow I$:

1. $I = [x_0, x_1] \cup [x_1, x_2] \cup \dots \cup [x_{n-1}, x_n]$, $x_0 = 0$, $x_n = 1$.
2. $I = [y_0, y_1] \cup [y_1, y_2] \cup \dots \cup [y_{n-1}, y_n]$, $y_0 = 0$, $y_n = 1$.
3. $\sigma \in S_n$ is a permutation of a set of n elements.
4. Transformation T maps (x_i, x_{i+1}) onto $(y_{\sigma(i)}, y_{\sigma(i)+1})$ in a continuous and bijective manner.

Then we demonstrate that the case of piecewise-continuous transformation can be reduced to interval exchange transformation.

WE construct the piecewise-continuous transformation T step by step. at the first iteration, we partition the interval into arbitrary subintervals which correspond to appropriate symbols. To construct the mapping, it is sufficient to determine the trajectory of these points and then, for reasons of recurrence, extend it by continuity to the entire interval.

Notation. By virtue of continuity and bijectivity, the mappings on the intervals are monotonic functions. We shall consider two cases:

1. All mappings of the intervals are increasing functions. Such transformation is said to be *orientation-preserving*.
2. There are both increasing and decreasing mappings of the intervals. In this case, we say that transformation *does not preserve orientation*.

Let us partition the segment $I = [0, 1]$ into $n = \text{Card } A$ arbitrary intervals, which will be regarded as characteristic sets for the symbols of alphabet A : $[0, 1] = I_{a_1} \cup I_{a_2} \cup \dots \cup I_{a_n}$.

The correspondence between the intervals and the symbols is defined by Proposition 3.5 below.

Let us assume that the mapping T is continuous on each set I_{a_i} , i.e., mapping T can be discontinuous only at the endpoints of the characteristic sets. The intervals of the characteristic sets (or their images) that have a common point are said to be *adjacent*.

Remark. We can always enlarge the alphabet in such a way that characteristic sets in the extended alphabet be organized exactly as described above. The graphs of the k -words in the original alphabet then correspond to 1-graphs of the extended alphabet and the evolutions of graphs coincide starting from this moment.

One can directly verify the following assertion.

Proposition 3.4 *The images of two adjacent sets under transformations T and $T^{(-1)}$ either are adjacent, or cannot cover the entire interval.*

Proposition 3.5 *The partition intervals can be put in correspondence with the symbols of the alphabet in such a way that, if w is a special right 1-word and wa_i , wa_j are subwords, then sets I_{a_i} and I_{a_j} are adjacent. Similarly, if w is a special left 1-word and $a_k w$, $a_l w$ are subwords, then sets I_{a_k} and I_{a_l} are also adjacent.*

Proof. Consider an arbitrary crotch in the 1-graph. Suppose that arcs that go out of this crotch lead to vertices that correspond to symbols a_i and a_j . Then we set $I_{a_i} = [x_0, x_1]$, $I_{a_j} = [x_1, x_2]$,

Since the characteristic sets are intervals, there is only one order relation that can be introduced on them (namely, $I_{a_i} < I_{a_j}$ if $x_i < x_j$); the same order relation can be introduced on their images. If a pair of characteristic sets reverse their order under transformation T , then we say that, on this pair, the transformation changes orientation.

Consider the images of the intervals under the mapping T^{-1} . It is clear that, if symbol a_i is not a right special 1-word and is inevitably followed by symbol a_j , then $I_{a_i} \subset T^{-1}(I_{a_j})$; if it is not a left special 1-word and is always preceded by symbol a_k , then $T^{-1}(I_{a_i}) \subset I_{a_k}$.

In the case where symbol a_i is a right special 1-word and can be followed by symbols a_j and a_k , we have $I_{a_i} \subset T^{-1}(I_{a_j} \cup I_{a_k})$; if it is a left special one, then $T^{-1}(I_{a_i}) \subset I_{a_j} \cup I_{a_k}$.

Denote the set of the images of the interval ends under the mapping T^{-n} by I^n and denote the set of the ends of the intervals of characteristic sets by I^0 , i.e., $I^0 = \{x_0, x_1, \dots, x_n\}$.

As follows from considerations in Section 2.4, the set that corresponds to the word $w = w_1 w_2 \dots w_n$ is $I_w = I_{w_n} \cap T^{-1}(I_{w_{n-1}}) \cap \dots \cap T^{-(n-1)}(I_{w_1})$; accordingly, the set of boundary points of the sets that correspond to words of length n is $I^0 \cup I^1 \cup \dots \cup I^{(n-1)}$.

If a right special word is not a bispecial one (which means that it is not at the same time a left special one), then the location of the point that partitions this characteristic set does not matter and it can be chosen arbitrarily.

In the case of orientation-preserving transformation, the partition should be in agreement with orientation-preserving rules.

In the case where transformation does not preserve orientation, it is necessary that, in the process of evolution, the number of “breaks” inside the intervals being mapped be finite.

Thus, we can define the images of points on a certain subset $N \subset I$. It follows from the construction that there exist intervals $I_k = (x_k, x_{k+1})$ such that, inside these intervals, our transformation is monotonic. We can always extend it by continuity to a mapping of the interval into itself. The resulting piecewise-continuous transformation is just what we looked for. Let us denote it by T . Note that the initial point whose evolution is the desired word $W = \{w_n\}$ is the point of intersection for the sequence of nested intervals which correspond to prefixes $w_0, w_0 w_1, w_0 w_1 w_2, \dots$

Thus, we have proved

Theorem 3.6 *For a recurrent word W to be generated by piecewise-continuous orientation-preserving transformation, it is necessary and sufficient that the word be provided with asymptotically correct oriented evolution of labeled Rauzy graphs.*

In the case of orientation-changing transformation, we have

Theorem 3.7 *For a recurrent word W to be generated by piecewise-continuous transformation, it is necessary and sufficient that the word be provided with asymptotically correct evolution of labeled Rauzy graphs.*

4 Equivalence of the set of uniformly recurrent words generated by piecewise–continuous transformation to the set of words generated by interval exchange transformation

First, let us pass to the dynamics in which almost all points (in the sense of the Lebesgue measure) have distinct essential evolutions. Consider the essential evolutions of the points under transformation T .

Consider a piecewise-continuous transformation of intervals. The theorem of existence of invariant measure (see [4]) yields that any mapping of a compact set has invariant probabilistic measure. Therefore, the mapping T has invariant measure μ and we can introduce a new semimetric $d(x_1, x_2) = \mu((x_1, x_2))$ on the segment. Note that the inequality $\mu(x_1, x_2) > 0$ does not imply that the points have different essential evolutions, since the mapping constructed can be discontinuous. Moreover we need chose a suitable measure.

The partition U_1, \dots, U_n of interval is called a *pure partition* if following conditions hold: 1) for any a_i characteristic set U_i is convex, i.e. U_i is closed, semienclosed or open interval. 2) if two points x_1, x_2 has a same color and the interval (x_1, x_2) contain a break point then images $T(x_1)$ and $T(x_2)$ has a different colors.

Let U_1, U_2, \dots, U_n be a partition into characteristic sets. The partition V_1, V_2, \dots, V_m is called *subpartition* if each characteristic set U_i is union of sets V_j : $U_i = V_{i_1} \cup V_{i_2} \cup V_{i_j}$.

Let W be an evolution for some partition and $W' = \{w'_i\}$ be an evolution for its subpartition. It is clear that W raise from W' by gluing of symbols. It is easy to see that every partition has a pure subpartition.

Let W would be an uniformly recurrent word, corresponding some evolution with partition \mathcal{U} , \widehat{W} is a word, corresponding the evolution with the same initial point and subpartition \mathcal{U}' . Gluing morphism of alphabets give us a natural morphism π of the words $\pi : \widehat{W} \rightarrow W$. Logically, \widehat{W} may not necessary uniformly recurrent, but there is an u.r. \widehat{U} such that $\widehat{U} \preceq \widehat{W}$. Then $U = \pi(\widehat{U}) \preceq W = \pi(\widehat{W})$ and hence u.r via theorem 2.4. In the corresponding to \widehat{U} closed orbit would be point corresponding an evolution with projection W . Indeed, let w be an arbitrary subword of W , it occurs in any subword of W of length $\geq k(w)$. Hence in any subword of \widehat{U} of length $\geq k(w)$ there exist a subword \hat{w} such that $\pi(\hat{w}) = w$. Hence for any surrounding of zero position of W there exist $\widehat{V} \equiv \widehat{V}$ such that $\pi(\widehat{V})$ has the same surrounding. It remains to use compactness argument. Corresponding word \widehat{W}' will be u.r. In the sequel we consider only pure partitions.

Proposition 4.1 *Suppose that points $x_1 < x_2$ have the same evolution and and the partition is pure. Then any point of the interval (x_1, x_2) has the same essential evolution.*

Proof. The fact that x_1 and x_2 have the same evolution yields that $T(x_1)$ and $T(x_2)$ have the same evolution as well and the image of the interval (x_1, x_2) is the interval $(T(x_1), T(x_2))$. \square

Let W be an nonperiodic uniformly recurrent word generated by a piecewise–continuous transformation of the interval. Consider the space of words with shift operator. For W there exist minimal invariant set N_W (see [1]). Every point $V \in N_W$ correspond set of points of system of intervals with corresponding essential evolution. On the space N_W there exist invariant (for shift operator) probabilistic measure ν ([4]). This measure

induce on interval system invariant probabilistic measure μ by the natural way. The more detailed description of this measure μ is follows. If I is an interval, corresponding to the word u , and N is the set of points of closed orbit of W having subword u on given position, then $\mu(I) = \nu(N)$. If I has no intersection with any such intervals, then $\mu(I) = 0$.

The measure μ is invariant and induces a semimetric. (Length of any segment is equal its measure.)

Consider topology that glue point with zero distance and construct corresponding factor-dynamics. Obtaining map is piecewise-isometric, i.e it is interval exchange transformations. Every such glued interval is contained in maximal glued interval that the number of such maximal intervals is countable and their common measure is zero. Hence for almost every point (in the sense of μ) of our compact M has orbit without intersection of any such glued interval. We have constructed piecewise continuous transformation of system M' generating an u.r. word, W' which is equivalent to W , and hence, via compactness argument, M' has a point whose essential evolution is W , because if there is a word with essential evolution W' , then any equivalent word can be observed.

We have proved

Theorem 4.2 *Suppose that the word W is generated by a piecewise-continuous transformation of the interval. Then there exists a word W' which is equivalent to the given one and is generated by interval exchange transformation.*

By theorem, this interval exchange transformation involves *all* words that are equivalent to W' including the word W (in the sense of the essential evolution). Thus, we have proved the following assertion.

Theorem 4.3 *For a recurrent word W to be generated by a piecewise-continuous orientation-preserving transformation, it is necessary and sufficient that the word be provided with asymptotically correct oriented evolution of labeled Rauzy graphs.*

In the case of orientation-changing transformation, we have proved the following.

Theorem 4.4 *For a recurrent word W to be generated by a piecewise-continuous orientation-changing transformation, it is necessary and sufficient that the word be provided with asymptotically correct evolution of labeled Rauzy graphs.*

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